

Vogan Diagrams of Twisted Affine Kac-Moody Lie Algebras

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Abstract

A Vogan diagram is a Dynkin diagram of a Kac-Moody Lie algebra of finite or affine type overlayed with additional structures. This paper develops the theory of Vogan diagrams for “almost compact” real forms of indecomposable twisted affine Kac-Moody Lie algebras and shows that equivalence classes of Vogan diagrams correspond to isomorphism classes of almost compact real forms of twisted affine Kac-Moody Lie algebras as given by H. Ben Messaoud and G. Rousseau in the paper “Classification des formes réelles presque compactes des algèbres de Kac Moody affines, J. Algebra 267”.

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1 Introduction

The classification of finite dimensional real simple Lie algebras has been a classical problem. In 1914, Élie Cartan classified the simple Lie algebras over the reals for the first time. A number of subsequent simplifications of the proof followed, until in 1996, using the theory of Vogan diagrams, A. W. Knap derived a quick proof of Élie Cartan’s classification in [8].

The Kac-Moody Lie algebras are an infinite-dimensional generalization of the semisimple Lie algebras via the Cartan matrix and generators. The real forms of complex affine Lie algebras are of two kinds, “almost split” and “almost compact.” V. Back et. al. classified

the almost split real forms of the affine Kac-Moody Lie algebra in [4] and H. Ben Messaoud and G. Rousseau gave a classification of the almost compact real forms in [12]. Work towards developing the theory of Vogan diagrams for the real forms of non-twisted affine Kac-Moody Lie algebras was done by P. Batra in [1, 2]. In the present paper the theory of Vogan diagrams for twisted affine Kac-Moody Lie algebras is developed.

As introduced in [7], a Vogan diagram is a Dynkin diagram of a Lie algebra with a diagram involution, such that the vertices fixed by the involution are either painted or unpainted depending on whether they are noncompact or compact. An important result in the theory of Vogan diagrams for real simple Lie algebras states that any Vogan diagram can be transformed, by changing the ordering of its base, into a diagram which has at most one noncompact imaginary root and that root occurs at most twice in the largest root of that simple Lie algebra. Since in the case of affine Kac-Moody algebras, changing the order does not give a Vogan diagram with at most one shaded root, therefore a notion of equivalence of Vogan diagrams for non-twisted affine Kac-Moody Lie algebras was introduced in [1]. In the present paper we modify the definition of the Vogan diagrams for the twisted affine Kac-Moody Lie algebras. In addition to the structural information already superimposed, a Vogan diagram now contains numerical labels on the vertices of the underlying Dynkin diagram as given in Figure 1. The classification of the almost compact real forms of affine Kac-Moody Lie algebras as given in [12], prompts the definition of suitable equivalence relations among the Vogan diagrams for twisted affine Kac-Moody Lie algebras. With respect to this equivalence relation we prove the following result.

Theorem. Let \mathfrak{g} be a twisted affine Kac-Moody Lie algebra. Then

1. Two almost compact real forms of \mathfrak{g} having equivalent Vogan diagrams are isomorphic.
2. Every abstract Vogan diagram for \mathfrak{g} , represents an almost compact real form of \mathfrak{g} .

The analogues of these results for the non-twisted Kac-Moody Lie algebras were proved by P. Batra in [1, Theorem 5.2] and [2, Theorem 5.2] respectively. Owing to the difference in the structural realizations of the non-twisted and twisted affine Kac-Moody Lie algebras, the methods used in [1, 2] prove insufficient to yield the main theorems for the twisted affine Kac-Moody Lie algebras. This difficulty is resolved by using the notion of “adapted realization” of an affine Kac-Moody Lie algebra as introduced in [12].

It is a fairly easy matter to work out representatives of the equivalence classes of Vogan diagrams for the twisted affine Kac-Moody Lie algebras. These have been listed in Figures

2 and 3. A match in the count of the non-equivalent Vogan diagrams and the count of non-isomorphic almost compact real forms as given in [12], seems to suggest the existence of a bijective correspondence between the equivalence classes of Vogan diagrams and the isomorphism classes of almost compact real forms of twisted affine Kac-Moody Lie algebras.

This paper is organized as follows: Section 2 reviews known facts about (complex) indecomposable twisted affine Kac-Moody Lie algebras \mathfrak{g} . In Section 3, the automorphisms and real forms of \mathfrak{g} are discussed, certain results from [12], which allow simple proofs of the main theorems are recalled and some properties of the Cartan subalgebras of the almost compact real forms of \mathfrak{g} are studied. In Section 4, the Vogan diagrams are introduced, their equivalence relations defined and the main theorems are stated and proved. In Figure 2 and 3, the non-equivalent Vogan diagrams for the twisted affine Kac-Moody Lie algebras are given.

Notational Convention. The complexification $\mathfrak{m} \otimes \mathbb{C}$ of a real Lie algebra \mathfrak{m} will be denoted by $\mathfrak{m}_{\mathbb{C}}$. Given a finite order automorphism ϕ of a Lie algebra \mathfrak{L} , we shall denote by \mathfrak{L}^{ϕ} , the fixed point subalgebra $\{x \in \mathfrak{L} \mid \phi(x) = x\}$ of \mathfrak{L} . By abuse of notation we shall denote $\mathbb{Z} \cap [m, n]$ by $[m, n]$ for all $m, n \in \mathbb{Z}$. For all integers n , ε_n will denote the n^{th} root of unity.

2 Kac-Moody Lie algebras

2.1. Let \mathfrak{g} be an affine Kac-Moody Lie algebra over the complex field \mathbb{C} . There exists a generalized Cartan matrix $A = (a_{i,j})_{i,j \in [0,l]}$ such that $\mathfrak{g} = \mathfrak{g}(A)$ is generated by the Cartan subalgebra \mathfrak{h} and the elements e_i, f_i for $i \in [0, l]$ (cf. [6, Chapter 1]). We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha})$, where $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ denotes the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\pi = \{\alpha_i \mid i \in [0, l]\}$ be the standard base of Δ ; $\Delta_+ = \Delta \cap (\bigoplus_{i \in [0, l]} \mathbb{N}\alpha_i)$ the set of positive roots and $\Delta_- = -\Delta_+$ the set of negative roots of \mathfrak{g} . The coroots $(\alpha_i^{\vee})_{i \in [0, l]} \subset \mathfrak{h}$ are such that $a_{i,j} = \alpha_j(\alpha_i^{\vee})$ for $i, j \in [0, l]$. Let W denote the Weyl group of \mathfrak{g} . $\alpha \in \Delta$ is said to be a real root, if α is W -conjugate to a root in π and the set of real roots is denoted by Δ^{re} . The elements of $\Delta^{im} = \Delta \setminus \Delta^{re}$ are called the imaginary roots of \mathfrak{g} .

2.2. Realization of a Kac-Moody Lie algebra : Let \mathfrak{g} be a finite dimensional simple Lie algebra over \mathbb{C} , μ a k -order automorphism of \mathfrak{g} for $k < \infty$, $\varepsilon_k = e^{\frac{2i\pi}{k}}$, a primitive k^{th} root of unity and (\cdot, \cdot) a nondegenerate, invariant, symmetric bilinear form on \mathfrak{g} . For $j \in \mathbb{Z}_k$, let $\mathfrak{g}_j = \{X \in \mathfrak{g} \mid \mu(X) = \varepsilon_k^j X\}$; then $\mathfrak{g} = \bigoplus_{j=0}^{k-1} \mathfrak{g}_j$. By [6], $(\mathfrak{g})^{\mu} = \mathring{\mathfrak{g}} = \mathfrak{g}_0$ is a simple finite dimensional Lie algebra over \mathbb{C} . If $\mathring{\mathfrak{h}}$ denotes the Cartan subalgebra of \mathfrak{g} , then $\mathring{\mathfrak{h}} = \mathring{\mathfrak{h}} \cap \mathfrak{g}_0$ is the Cartan

subalgebra of $\dot{\mathfrak{g}}_0$. We denote by \mathfrak{g} the infinite dimensional Lie algebra:

$$\mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k) = \left(\bigoplus_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_{(j \bmod k)} \otimes t^j \right) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The Lie algebra structure on \mathfrak{g} is such that c is the canonical central element and

$$[x \otimes t^m + \lambda d, y \otimes t^n + \lambda_1 d] = ([x, y] \otimes t^{m+n} + \lambda n y \otimes t^n - \lambda_1 m x \otimes t^m) + m \delta_{m, -n}(x, y)c,$$

where $x, y \in \dot{\mathfrak{g}}$, $\lambda, \lambda_1 \in \mathbb{C}$. The element d acts diagonally on \mathfrak{g} with integer eigenvalues and induces \mathbb{Z} -gradation on $\mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$. The Lie algebra $\mathfrak{l}(\dot{\mathfrak{g}}, Id, 1)$ with $\mu = Id$ denotes a non-twisted affine Kac-Moody Lie algebra and $\mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$ for $\mu \neq Id$ and $k = 2$ or 3 denotes a twisted affine Kac-Moody Lie algebra. Clearly $\mathfrak{g}'' = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)'' = \bigoplus_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_{(j \bmod k)} \otimes t^j$ is the fixed point set of the automorphism $\tilde{\mu}$ of $\mathfrak{l}(\dot{\mathfrak{g}}, Id, 1)'' = \dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$ defined by:

$$\tilde{\mu}(x \otimes t^j) = (\epsilon_k)^j \mu(x) \otimes t^j, \quad \text{for } j \in \mathbb{Z}, x \in \dot{\mathfrak{g}}. \quad (2.1)$$

$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \otimes 1 \oplus \mathbb{C}c \oplus \mathbb{C}d$ is the standard Cartan subalgebra of \mathfrak{g} .

2.3. Let $(\dot{e}_i, \dot{f}_i)_{i=1, \dots, n}$ be a system of Chevalley generators of $\dot{\mathfrak{g}}$. The simple coroots, $\dot{\alpha}_i^\vee = [\dot{e}_i, \dot{f}_i]$ for $i = 1, \dots, n$ form a base of $\dot{\mathfrak{h}}$. Let $\dot{\omega}$ be the Cartan involution of $\dot{\mathfrak{g}}$ given by $\dot{\omega}(\dot{e}_i) = -\dot{f}_i$, $\dot{\omega}|_{\dot{\mathfrak{h}}} = -Id$ and $\dot{\omega}^2 = Id$. The simple roots in the base $\dot{\pi}$ of $\dot{\mathfrak{g}}$ are enumerated in a manner such that a system of representatives of the μ -orbits of $\{1, \dots, n\}$ is $\{1, \dots, l\}$. Since order of $\mu(\neq Id)$ is $k(=2 \text{ or } 3)$, therefore for any $i \in \{1, \dots, n\}$, the cardinality n_i of the μ -orbit of $\dot{\alpha}_i^\vee$ is either 1 or k . Correspondingly, $e_i = \dot{e}_i + \dots + \mu^{n_i-1}(\dot{e}_i)$ except in the case of $A_{2l}^{(2)}$ where $e_l = \sqrt{2}(\dot{e}_l + \dot{e}_{l+1})$; $f_i = \overset{\circ}{\omega}(\dot{e}_i)$; $\alpha_i^\vee = [e_i, f_i]$. Hence, $(e_i, f_i)_{i=1, \dots, l}$ is a system of Chevalley generators of the simple Lie algebra $\overset{\circ}{\mathfrak{g}}$. Let $\theta_0 \in (\overset{\circ}{\mathfrak{h}})^*$ be the highest weight of the irreducible $\overset{\circ}{\mathfrak{g}}$ -module $\overset{\circ}{\mathfrak{g}}_1$. Choose $E_0 \in (\overset{\circ}{\mathfrak{g}}_1)_{-\theta_0}$ and put $F_0 = -\dot{\omega}(E_0)$, $e_0 = E_0 \otimes t$, $f_0 = F_0 \otimes t^{-1}$ and $\alpha_0^\vee = [e_0, f_0]$. Then, $\{\mathfrak{h}, e_i, f_i, i \in [0, l]\}$ is a system of generators of $\mathfrak{g}(A)$, the Lie algebra associated to a generalized Cartan matrix A (cf. [6, Theorem 8.3]).

A compact form $\mathfrak{u}(A)$ of $\mathfrak{g}(A)$ is defined as the fixed point set of $\mathfrak{g}(A)$ under the compact involution ω defined on $\dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$ as follows: $\omega(x \otimes t^j) = \dot{\omega}(x) \otimes t^{-j}$.

2.4. A graph $S(A)$ called the Dynkin diagram of A can be associated to a generalized Cartan matrix (GCM) A as explained in [6]. A Dynkin diagram $S(\overset{\circ}{A})$ is obtained from $S(A)$ by removing the 0^{th} vertex. The corresponding GCM $\overset{\circ}{A}$ is of finite type and $\mathfrak{g}(\overset{\circ}{A}) = \overset{\circ}{\mathfrak{g}}$ is a simple finite dimensional Lie algebra. It is clear that A is indecomposable if and only if $S(A)$ is a

Type Aff k	\mathfrak{g}	Dynkin diagram	$\overset{\circ}{\mathfrak{g}}$
2	$A_2^{(2)}$	$\overset{1}{\underset{\alpha_0}{\circ}} = \overset{2}{\underset{\alpha_1}{\circ}}$	A_1
2	$A_{2l}^{(2)} \ (l \geq 2)$	$\overset{1}{\underset{\alpha_0}{\circ}} \Rightarrow \overset{2}{\underset{\alpha_1}{\circ}} - \dots - \overset{2}{\underset{\alpha_{l-1}}{\circ}} \Rightarrow \overset{2}{\underset{\alpha_l}{\circ}}$	B_l
2	$A_{2l-1}^{(2)} \ (l \geq 3)$	$\overset{1}{\underset{\alpha_0}{\circ}} \downarrow$ $\overset{1}{\underset{\alpha_1}{\circ}} - \overset{2}{\underset{\alpha_2}{\circ}} - \overset{2}{\underset{\alpha_3}{\circ}} - \dots - \overset{2}{\underset{\alpha_{l-1}}{\circ}} \Leftarrow \overset{1}{\underset{\alpha_l}{\circ}}$	C_l
2	$D_{l+1}^{(2)} \ (l \geq 2)$	$\overset{1}{\underset{\alpha_0}{\circ}} \Leftarrow \overset{1}{\underset{\alpha_1}{\circ}} - \dots - \overset{1}{\underset{\alpha_{l-1}}{\circ}} \Rightarrow \overset{1}{\underset{\alpha_l}{\circ}}$	B_l
2	$E_6^{(2)}$	$\overset{1}{\underset{\alpha_0}{\circ}} - \overset{2}{\underset{\alpha_1}{\circ}} - \overset{3}{\underset{\alpha_2}{\circ}} \Leftarrow \overset{2}{\underset{\alpha_3}{\circ}} - \overset{1}{\underset{\alpha_4}{\circ}}$	F_4
3	D_4^3	$\overset{1}{\underset{\alpha_2}{\circ}} - \overset{2}{\underset{\alpha_1}{\circ}} \Leftarrow \overset{1}{\underset{\alpha_2}{\circ}}$	G_2

Figure 1: Dynkin diagrams of the twisted affine Kac-Moody Lie algebras.

connected graph. The matrix A is determined by the Dynkin diagram and the enumeration of its vertices. Figure 1 gives the Dynkin diagrams of the twisted affine Kac-Moody Lie algebras. The enumeration of the generators is the same as in [6] except in the case $A_{2l}^{(2)}$, where the enumeration is reversed and the case $A_2^{(2)}$ where the Dynkin diagram and enumeration has been taken from [9, §3.5].

2.5. Let $\overset{\circ}{\Delta}$ be the root system of $\overset{\circ}{\mathfrak{g}}$. Let $\overset{\circ}{\pi} = (\alpha_i)_{i \in [1, l]}$ be a base of $(\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{h}})$ and $(\overset{\circ}{p}_i)_{i \in [1, l]}$ be the corresponding dual basis in $\overset{\circ}{\mathfrak{h}}$. Denote by $\overset{\circ}{\Delta}_s$, $\overset{\circ}{\Delta}_l$ and $\overset{\circ}{\Delta}_+$ the short, long and positive roots of $\overset{\circ}{\mathfrak{g}}$. Define an element $\delta \in \mathfrak{h}^*$ by putting $\delta(d) = 1$ and $\delta(\overset{\circ}{\mathfrak{h}} + \mathbb{C}c) = 0$. Recall from [6, Proposition 6.3] the following description of Δ^{re} for \mathfrak{g} not of type $A_{2l}^{(2)}$:

$$\Delta^{re} = \{ \alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta}_s, n \in \mathbb{Z} \} \cup \{ \alpha + nk\delta \mid \alpha \in \overset{\circ}{\Delta}_l, n \in \mathbb{Z} \}, \quad \text{if } k = 2 \text{ or } 3.$$

With respect to the enumeration of the simple roots of $A_{2l}^{(2)}$ as given in Figure 1, similar calculations as in [6, Proposition 6.3] show that for \mathfrak{g} of type $A_{2l}^{(2)}$, $\Delta^{re} = \Delta_s^{re} \cup \Delta_m^{re} \cup \Delta_l^{re}$ where,

$$\begin{aligned} \Delta_s^{re} &= \{ \tfrac{1}{4}(2\alpha + (4n - k)\delta) \mid \alpha \in \overset{\circ}{\Delta}_l, n, k \in \mathbb{Z}, 1 \leq k \leq 3 \}, \\ \Delta_m^{re} &= \{ \tfrac{1}{2}(2\alpha + (2n - 1)\delta) \mid \alpha \in \overset{\circ}{\Delta}_l, n \in \mathbb{Z} \}, \\ \Delta_l^{re} &= \{ 2\alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta}_l, n \in \mathbb{Z} \}. \end{aligned}$$

Let $\Delta_+^{re} = \{\alpha \in \Delta^{re} \text{ with } n > 0\} \cup \overset{\circ}{\Delta}_+$. By [6, §6.4, Proposition 6.4], $a_0\alpha_0 = \delta - \theta$ with $\theta \in (\overset{\circ}{\Delta}_+)_s$ for \mathfrak{g} of type Aff 2 or 3 and not of type $A_{2l}^{(2)}$ and $\alpha_0 = \delta - 2\theta$, with $\theta \in (\overset{\circ}{\Delta}_+)_l$ for \mathfrak{g} of type $A_{2l}^{(2)}$. If a_0, a_1, \dots, a_l are the numerical labels of $S(A)$ as in Figure 1, then the element $\delta \in \mathfrak{h}^*$ is defined as, $\delta = \sum_{i=0}^l a_i \alpha_i$ and we have,

$$\Delta^{im} = \{\pm\delta, \pm 2\delta, \dots\}, \quad \Delta_+^{im} = \{\delta, 2\delta, \dots\}.$$

The set $\Pi = (\alpha_i)_{i=0,1,\dots,l}$ is a base of Δ . Setting $p_0 = d$ and $p_i = \overset{\circ}{p}_i + a_i d$ for $i = 1, \dots, l$, we obtain a family $(p_i)_{i \in [0,l]} \subset \mathfrak{h}$ satisfying $\alpha_j(p_i) = \delta_{i,j}$ for $i, j \in [0, l]$.

Let \mathfrak{g}_γ be the root space of $\gamma \in \Delta$. For $\gamma \in \Delta^{re}$, $\dim \mathfrak{g}_\gamma = 1$.

For a root $\dot{\alpha}$ of $\dot{\mathfrak{g}}$, if $\dot{e}_{\pm\dot{\alpha}} \in \dot{\mathfrak{g}}_{\dot{\alpha}}$ is such that the \mathbb{C} -span of $\{\dot{e}_{\dot{\alpha}}, \dot{e}_{-\dot{\alpha}}, H_{\dot{\alpha}} = [\dot{e}_{\dot{\alpha}}, \dot{e}_{-\dot{\alpha}}]\}$ is isomorphic to \mathfrak{sl}_2 , then given $\alpha + nk\delta \in \Delta^{re}$ with $\alpha \in \overset{\circ}{\Delta}_l$, $E_{\pm\alpha, \pm ks} = e_{\pm\alpha} \otimes t^{\pm ks} \in \mathfrak{g}_{\pm(\alpha+ks)}$ and $H_\alpha \in \overset{\circ}{\mathfrak{h}}$ can be suitably chosen such that the \mathbb{C} -span of $\{E_{\alpha, ks}, H_\alpha, E_{-\alpha, -ks}\}$ is isomorphic to \mathfrak{sl}_2 . If $\alpha \in \overset{\circ}{\Delta}_s$, then there exists $\dot{\alpha} \in \dot{\mathfrak{g}}$ such that for $j \in \mathbb{Z}_k$ one can choose $E_{\pm\alpha, \pm ks+j} = (\varepsilon_k^j e_{\pm\mu(\dot{\alpha})} + \dots + \varepsilon_k^{jk} e_{\pm\mu^k(\dot{\alpha})}) \otimes t^{\pm ks+j} \in \mathfrak{g}_{\pm(\alpha+ks+j)}$ and $h_\alpha^j = \varepsilon_k^j H_{\mu(\dot{\alpha})} + \dots + \varepsilon_k^{jk} H_{\mu^k(\dot{\alpha})} \in \dot{\mathfrak{h}}_j$ such that the \mathbb{C} -span of $\{E_{\alpha, ks+j}, h_\alpha^j, E_{-\alpha, -ks-j}\}$ is isomorphic to \mathfrak{sl}_2 .

2.6. Let $\overset{\circ}{W}$ be the Weyl group of $\overset{\circ}{\mathfrak{g}}$ generated by the reflections $(r_i)_{i \in [1,l]}$. Let T be the group of translations. By [6, Proposition 6.5], $W = \overset{\circ}{W} \ltimes T$ is the Weyl of \mathfrak{g} . Since $w(\delta) = \delta$ for all $w \in W$, it follows from [6, §2.7 and Proposition 3.12(b)] that given two positive root systems Δ_+ and Δ'_+ of a compact form $\mathfrak{u}(A)$ of $\mathfrak{g}(A)$, there exists $s \in \overset{\circ}{W}$ such that $s.\Delta'_+ = \Delta_+$. But for $\alpha \in \overset{\circ}{\Delta}$, the generators r_α of $\overset{\circ}{W}$ are interior automorphisms (cf. [12, Lemma 5.3]). Hence given two positive root systems Δ_+ and Δ'_+ of $\mathfrak{u}(A)$ there exists $s \in \text{Int}(\mathfrak{u}(A))$ such that $s.\Delta'_+ = \Delta_+$.

3 Automorphisms and Real forms of \mathfrak{g}

3.1. Define a group G acting on \mathfrak{g} by the adjoint representation $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$. It is generated by the subgroups U_α , for $\alpha \in \Delta^{re}$, which are isomorphic to the additive groups \mathfrak{g}_α by an isomorphism \exp such that $\text{Ad} \circ \exp = \exp \circ \text{ad}$.

A Cartan subalgebra (CSA) of \mathfrak{g} is a maximal $\text{ad}_{\mathfrak{g}}$ -diagonalizable Lie subalgebra. The CSA's are all conjugate by G . A Borel subalgebra (BSA) is a completely solvable maximal subalgebra of \mathfrak{g} . $\mathfrak{b}^+ = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha)$ and $\mathfrak{b}^- = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha)$ are respectively called the

positive and negative standard BSA's. The subalgebras \mathfrak{b}^+ and \mathfrak{b}^- are not conjugate by G . All the BSA's conjugate to \mathfrak{b}^+ (respectively \mathfrak{b}^-) are said to be positive (respectively negative). If \mathfrak{g} is indecomposable, all BSAs are either positive or negative.

An automorphism (linear or semi-linear) of \mathfrak{g} acts in a compatible manner to Ad on G and hence transforms two conjugate BSAs to two conjugate BSAs; it is said to be of *first type* (respectively *second type*) if it transforms a positive BSA to positive (respectively negative) BSA. If \mathfrak{g} is indecomposable, all automorphisms are either of first or second type.

3.2. Automorphisms of \mathfrak{g} : By [11], the group of automorphisms of \mathfrak{g} is given by

$$\text{Aut}(\mathfrak{g}) = [(\{1, \omega\} \times \text{Aut}(A)) \ltimes \text{Int}(\mathfrak{g})] \ltimes \text{Tr},$$

where ω is the Cartan involution of \mathfrak{g} , $\text{Aut}(A)$ is the group of permutations of $[0, l]$ such that $a_{\rho_i, \rho_j} = a_{ij}$ for $i, j \in I$, $\text{Int}(\mathfrak{g})$ is the set of interior automorphisms of \mathfrak{g} and $\text{Tr} = \text{Tr}(\mathfrak{g}, \mathfrak{g}', \mathfrak{c})$ is the group of transvections of \mathfrak{g} as defined in [11, 2.4].

Let \mathfrak{h} be a standard CSA of \mathfrak{g} . A group \tilde{H} is defined such that in the complex case, $\text{Ad}(\tilde{H}) = \exp \text{ad}(\mathfrak{h})$ (cf.[10]). The group $\text{Int}(\mathfrak{g}) = \text{Ad}(\tilde{H} \ltimes G)$ of interior automorphisms of \mathfrak{g} is the image of the semi-direct product of \tilde{H} and G . Its derived group is the adjoint group $\text{Ad}(G)$ (denoted by $\text{Int}(\mathfrak{g}')$). As G acts transitively on the Cartan subalgebra, the group $\text{Int}(\mathfrak{g}')$ does not depend on the choice of \mathfrak{h} .

Definition. Let $\text{Aut}_{\mathbb{R}}(\mathfrak{g})$ denote the group of automorphisms of \mathfrak{g} that are either \mathbb{C} -linear or semilinear (i.e., $\phi(\lambda x) = \bar{\lambda}\phi(x)$, $\forall \lambda \in \mathbb{C}, x \in \mathfrak{g}$). $\text{Aut}(\mathfrak{g})$ is an index 2 normal subgroup of $\text{Aut}_{\mathbb{R}}(\mathfrak{g})$.

A semi-involution of \mathfrak{g} is a semi-linear automorphism of order 2. For all semi-involutions σ' we have a decomposition, $\text{Aut}_{\mathbb{R}}(\mathfrak{g}) = \{1, \sigma'\} \ltimes \text{Aut}(\mathfrak{g})$. If σ' is a semi-involution of \mathfrak{g} , the real Lie algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\sigma'}$ is a real form of \mathfrak{g} , in the sense that there exists an isomorphism of the complex Lie algebras $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{C}} \mathbb{C}$ and \mathfrak{g} ; further, σ' is the conjugation of \mathfrak{g} with respect to $\mathfrak{g}_{\mathbb{R}}$. Thus there exists a bijective correspondence between the semi-involutions and real forms. The *standard normal(or split)* real form of \mathfrak{g} is the real Lie algebra generated by $e_i, f_i, \alpha_i^{\vee}$ and d . The corresponding semi-involution σ'_n is called the *normal semi-involution*. Note that σ'_n is the restriction of $\dot{\sigma}'_n \otimes \text{conj}$ on $\mathfrak{g}'' = (\dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}])^{\bar{\mu}}$, where $\dot{\sigma}'_n$ is the normal semi-involution of $\dot{\mathfrak{g}}$ and $\text{conj}(P(t)) = \bar{P}(t)$. σ'_n commutes with the standard Cartan involution ω .

The *standard Cartan semi-involution* ω' of \mathfrak{g} is the unique semi-involution of \mathfrak{g} such that $\omega'(e_i) = -f_i$, and $\omega'(d) = -d$. Hence $\omega' = \sigma'_n \omega = \omega \sigma'_n$. In the standard realization of \mathfrak{g} ,

ω' induces on \mathfrak{g}'' the restriction of $\dot{\omega} \otimes \iota'$, where $\dot{\omega}$ is the Cartan semi-involution of $\dot{\mathfrak{g}}$ and $\iota'(P(t)) = \bar{P}(t^{-1})$. All conjugates of ω' are called *Cartan semi-involutions (CSI)* or *compact semi-involutions*; these are semi-involutions of the second type. The corresponding real forms are called the *compact real forms*. It is clear that, for all affine Kac-Moody Lie algebras, there exists, upto a conjugation, a unique compact real form.

3.3. Real Forms of \mathfrak{g} : The real form corresponding to a semi-involution of first type (SI1) (respectively of second type (SI2)) is said to be *almost split* (respectively *almost compact*) real form. Upto a conjugation, a classification of the almost split real forms was given in [4] and a classification of the almost compact real forms was given in [12].

3.4. Cartan subalgebra of a real form of \mathfrak{g} : Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of the complex Lie algebra \mathfrak{g} . A Lie subalgebra \mathfrak{h}_0 of $\mathfrak{g}_{\mathbb{R}}$ is called the *Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$* if the complexification, $\mathfrak{h}_0 \otimes \mathbb{C}$ is a Cartan subalgebra of \mathfrak{g} .

3.5. Cartan Involutions: Let σ' be a SI2 of \mathfrak{g} , and let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\sigma'}$ be the corresponding almost compact real form. A CSI ϑ that commutes with σ' is said to be *adapted* to σ' or $\mathfrak{g}_{\mathbb{R}}$. The involution $\sigma = \sigma'\vartheta$ (respectively its restriction $\vartheta_{\mathbb{R}}$ to $\mathfrak{g}_{\mathbb{R}}$ or $\sigma'_{\mathbb{R}}$ to $\mathfrak{u} = \mathfrak{g}^{\vartheta}$) is said to be the Cartan involution of σ' (respectively of $\mathfrak{g}_{\mathbb{R}}$ or of \mathfrak{u}). The algebra of fixed points $\mathfrak{k} = \mathfrak{g}_{\mathbb{R}}^{\sigma} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{u} = \mathfrak{u}^{\sigma}$ is the *maximal compact subalgebra* of $\mathfrak{g}_{\mathbb{R}}$. $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$; $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ and \mathfrak{u} into eigenspaces of σ . A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is said to be *maximally compact* for σ' (or $\mathfrak{g}_{\mathbb{R}}$) if it is stable under σ' and if $-\sigma'$ stabilizes a base of $\Delta(\mathfrak{g}, \mathfrak{h})$.

Proposition 3.6. [11, Proposition 2.8] *Let σ' be a SI2 of \mathfrak{g} and let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\sigma'}$.*

1. *There exists CSI's ϑ adapted to σ' and maximally compact CSA's \mathfrak{h} for σ' .*
2. *For all maximally compact CSA's \mathfrak{h} for σ' , there exists a CSI ϑ adapted to σ' that stabilizes \mathfrak{h} , and it is unique upto an interior automorphism fixing \mathfrak{h} and commuting with σ' .*
3. *For all CSI ϑ adapted to σ' , there exists a maximally compact Cartan subalgebra for σ' stabilized by ϑ , and it is unique upto an interior automorphism commuting with σ' and ϑ .*

Proposition 3.7. [11, Proposition 2.9] *Consider:*

1. *σ , an involution of first type of \mathfrak{g} .*
2. *the pairs (σ', \mathfrak{h}) formed of a semi-involution of second type σ' , which is not Cartan, and a maximally compact Cartan sub-algebra for σ' .*
3. *the relation $(\sigma', \mathfrak{h}) \sim \sigma$ if and only if σ commutes with σ' , stabilizes \mathfrak{h} and is such that $\sigma\sigma'$ is a Cartan semi-involution of \mathfrak{g} .*

This relation induces a bijection between the conjugacy classes (under $\text{Int}_{Tr}(\mathfrak{g})$ or $\text{Aut}(\mathfrak{g})$) of involutions of the first type, σ and the pairs (σ', \mathfrak{h}) .

Note: With notations as above, if $(\sigma', \mathfrak{h}) \sim \sigma$, then we say that σ is adapted to σ' or $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\sigma'}$.

Notation. In what follows, $\dot{\mathfrak{s}}$ will denote a finite dimensional semisimple Lie algebra over \mathbb{C} .

3.8. Adapted Realization of \mathfrak{g} : Recall the following definitions from [12]:

Definition. Let σ be finite order automorphism of \mathfrak{g} of first type and \mathfrak{h} a maximally fixed Cartan subalgebra for σ . A realization of \mathfrak{g} on which σ preserves the \mathbb{Z} -gradation and for which \mathfrak{h} is the standard Cartan subalgebra is said to be *almost adapted to (σ, \mathfrak{h})* . An almost adapted realization $\mathfrak{l}(\dot{\mathfrak{s}}, \zeta, \varepsilon_m)$ for (σ, \mathfrak{h}) on which σ commutes with the translation map $\mathcal{T} : x \mapsto x \otimes t^m$, is said to be *adapted to σ* (respectively to (σ, \mathfrak{h})).

Definition. Let σ' be finite order automorphism of \mathfrak{g} of second type, \mathfrak{h} a maximally compact Cartan subalgebra for σ' and σ an involution of first kind associated to the pair (σ', \mathfrak{h}) . A realization $\mathfrak{l}(\dot{\mathfrak{s}}, \mu, \varepsilon_m)$ of \mathfrak{g} which is adapted to (σ, \mathfrak{h}) is said to be *adapted to $(\sigma', \sigma, \mathfrak{h})$* if there exists an involution $\dot{\sigma}$ and a semi-involution $\dot{\sigma}'$ of $\dot{\mathfrak{s}}$ commuting with μ such that $\dot{\sigma}\dot{\sigma}'$ is a Cartan semi-involution of $\dot{\mathfrak{s}}$, $\sigma = \dot{\sigma} \otimes 1$ and $\sigma' = \dot{\sigma}' \otimes \iota'$ on the realization $\mathfrak{l}''(\dot{\mathfrak{s}}, \mu, \varepsilon_m)$; and finally $\sigma'(c) = -c$ and $\sigma'(d) = -d$.

It is known from [12, Proposition 3.4, Theorem 3.5 and Proposition 3.9] that given a finite order automorphism σ of first type (respectively σ' of second type) of \mathfrak{g} , there exists realizations of \mathfrak{g} adapted to σ (respectively adapted to σ').

3.9. Let σ be an involution adapted to a semi-involution of second type σ' . Since σ is an involution of \mathfrak{g} of the first kind, by [3, Chapter II] upto an interior automorphism, either $\sigma = \rho H$ or $\sigma = H$, where ρ is a diagram automorphism of $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$ and H is an interior automorphism of \mathfrak{g} of the form $\exp i\pi \text{ad}(h_0)$, with $h_0 \in \mathfrak{h}_{\mathbb{Z}}^{\rho}$ where $\mathfrak{h}_{\mathbb{Z}}^{\rho} = \{x \in \mathfrak{h}^{\rho} \mid \alpha(x) \in \mathbb{Z}, \forall \alpha \in \Delta\}$. Further $h \in \mathfrak{h}_{\mathbb{Z}}$ can be written as $h = h' + \eta d$, with $h' \in (\dot{\mathfrak{g}})^{\mu}$ and $\eta \in \mathbb{Z}$.

Proposition 3.10. [12, Proposition 2.12] *With notations as above:*

1. If η is even or $k = 2$ (i.e., \mathfrak{g} is of type Aff 2) then the interior involution H respects the \mathbb{Z} -gradation of $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$, commutes with the map $\mathcal{T} : x \mapsto x \otimes t^k$ on \mathfrak{g}'' and there exists an interior involution \dot{H} of $\dot{\mathfrak{g}}$ commuting with μ such that $H = (\mu^n \dot{H}) \otimes 1$ on $\mathfrak{l}''(\dot{\mathfrak{g}}, \mu, \varepsilon_k) = \mathfrak{g}''$.
2. If η is odd and $k \neq 2$, then the interior involution H induces on \mathfrak{g}'' the automorphism $t^k \mapsto -t^k$ and H in this case acts on the adapted realization $\mathfrak{l}(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}, \zeta, \varepsilon_{2k})$ of \mathfrak{g} , with $\zeta(x, y) =$

$(y, \mu(x))$.

3. Each diagram automorphism of a twisted affine Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_m)$ is \mathbb{A} -linear, where \mathbb{A} is the algebra generated by the translation maps $\mathcal{T}^\pm : x \mapsto x \otimes t^{\pm m}$.

The following are some examples of realizations of $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$, $k = 2, 3$, adapted to $(\sigma, \sigma', \mathfrak{h})$.

Example 1. Let $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, -1)$ where $\dot{\mathfrak{g}}$ is of type A_{2l-1} .

i. Let $\sigma = H$, where H is an interior automorphism of $\dot{\mathfrak{g}}$. Then by Proposition 3.10(1), $\mathfrak{l}(\dot{\mathfrak{g}}, \mu, -1)$ is adapted to $(\sigma, \sigma', \mathfrak{h})$.

ii. Let $\sigma = \rho$, where ρ is a diagram automorphism of $A_{2l-1}^{(2)}$ such that $\rho(\alpha_0) = \alpha_1$, $\rho(\alpha_1) = \alpha_0$ and $\rho(\alpha_i) = \alpha_i$ for $i \in [2, l]$. In the given realization since $\sigma(E_0 \otimes t) = \rho(E_0 \otimes t) = e_1 \otimes 1$, σ clearly cannot be written in the form $\dot{\sigma} \otimes 1$ for any $\dot{\sigma} \in \text{Aut}(\dot{\mathfrak{g}})$. Hence using [12, 1.7], with $H_1 = \exp \frac{i\pi}{2} \text{ad}(\overset{\circ}{p}_1)$ we consider the realization $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_1, \varepsilon_4)$ of \mathfrak{g} which is isomorphic to $\mathfrak{l}(\dot{\mathfrak{g}}, \mu, -1)$ via the map $\psi : \mathfrak{l}(\dot{\mathfrak{g}}, \mu, -1) \mapsto \mathfrak{l}(\dot{\mathfrak{g}}, \mu H_1, \varepsilon_4)$ defined as follows:

$$\begin{aligned} \psi(x \otimes t^j) &= x \otimes t^{2j+N} + \delta_{j,0}(\overset{\circ}{p}_1, x)C; \quad \text{for } x \in \dot{\mathfrak{g}}_j \text{ such that } [\overset{\circ}{p}_1, x] = Nx, \\ \psi(c) &= 2C, \quad \psi(d) = (D - \overset{\circ}{p}_1)/2, \end{aligned}$$

where C and D respectively denote the central and gradation elements of $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_1, \varepsilon_4)$. Since $[\overset{\circ}{p}_1, E_0] = -E_0$ and $[\overset{\circ}{p}_1, e_1] = e_1$, it is easy to see that under the new realization $e_0 = E_0 \otimes t$ is mapped to $E_0 \otimes t$ and $e_1 = e_1 \otimes 1$ is mapped to $e_1 \otimes t$ and hence ρ can be written in the form $\dot{\sigma} \otimes 1$ where $\dot{\sigma} \in \text{Aut}(\dot{\mathfrak{g}})$ is an involution such that $\dot{\sigma}(E_0) = e_1$, and $\dot{\sigma}(e_i) = e_i$ for $i \in [2, l]$. Taking $\dot{\sigma}' = \dot{\sigma}\dot{\omega}$ it can be easily seen that $\dot{\sigma}'\mu H_1 = \mu H_1\dot{\sigma}'$ on $\dot{\mathfrak{g}}$. Hence the realization $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_1, \varepsilon_4)$ is adapted to $(\rho, \rho\omega', \mathfrak{h})$.

Example 2. Let $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_3)$ where $\dot{\mathfrak{g}}$ is of type D_4 .

i. Let $\sigma = \exp i\pi \text{ad}(p_1)$ or $\exp i\pi \text{ad}(\overset{\circ}{p}_2)$. Then by Proposition 3.10(1), $\mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_3)$ is adapted to $(\sigma, \sigma', \mathfrak{h})$.

ii. Let $\sigma = \exp i\pi \text{ad}(p_0)$. Then by Proposition 3.10(2), $\mathfrak{l}(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}, \zeta, \varepsilon_6)$ is an adapted realization of $\mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_3)$. It is known from [12] that for $\zeta \in \text{Aut}(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}})$ as defined above, $\phi_2 : \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_3) \mapsto \mathfrak{l}(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}, \zeta, \varepsilon_6)$ is defined by:

$$\phi_2(x \otimes t^j) = (x, \varepsilon_6 x) \otimes t^j, \quad \phi_2(c) = 2c_2, \quad \phi_2(d) = d_2,$$

gives an isomorphism between the two realizations of \mathfrak{g} . Here c_2 and d_2 respectively denote the central and gradation elements of $\mathfrak{l}(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}, \zeta, \varepsilon_6)$.

3.11. Let σ' be a semi-involution of second type of \mathfrak{g} and let $\mathfrak{g}_{\mathbb{R}} = (\mathfrak{g})^{\sigma'}$ be the corresponding almost compact real form of \mathfrak{g} . Let ω' is a Cartan semi-involution of \mathfrak{g} adapted to σ' . Then

$B_{\omega'}$ defined by $B_{\omega'}(X, Y) = -B(X, \omega'(Y))$ is a positive definite hermitian form on $\mathfrak{g}_{\mathbb{R}}$ and given $X, Y, Z \in \mathfrak{g}_{\mathbb{R}}$, we have:

$$\begin{aligned}
 B_{\omega'}(\text{ad } \sigma X(Y), Z) &= -B([\sigma X, Y], \omega' Z) \\
 &= B(Y, [\sigma X, \omega' Z]) \\
 &= B(Y, \omega'[\sigma' X, Z]) \\
 &= -B_{\omega'}(Y, \text{ad } \sigma' X(Z)).
 \end{aligned} \tag{3.1}$$

Since for all $X \in \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\sigma'}$, $\sigma' X = X$, therefore with respect to $B_{\omega'}$, we get

$$(\text{ad } \sigma X)^* = -\text{ad } \sigma' X = -\text{ad } X \quad \forall X \in \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{\sigma'}.$$

This implies that for $X \in \mathfrak{k} = \mathfrak{g}^{\sigma}$, $\text{ad } X$ is skew-symmetric with purely imaginary eigenvalues and for $X \in \mathfrak{p}$, $\text{ad } X$ is symmetric with real eigenvalues.

Let $\mathfrak{l}(\dot{\mathfrak{s}}, \mu, \varepsilon_m)$ be a realization of \mathfrak{g} adapted to $(\sigma', \sigma, \mathfrak{h})$ where $\dot{\mathfrak{s}}$ is a complex semi-simple Lie algebra. Let $\dot{\sigma}, \dot{\sigma}' \in \text{Aut}(\dot{\mathfrak{s}})$ such that $\sigma = \dot{\sigma} \otimes 1$ and $\sigma' = \dot{\sigma}' \otimes \iota'$. Let $\dot{\mathfrak{s}}_{\mathbb{R}} = \dot{\mathfrak{s}}^{\dot{\sigma}'}$ and $\dot{\mathfrak{s}}_{\mathbb{R}} = \dot{\mathfrak{k}} \oplus \dot{\mathfrak{p}}$ be its Cartan decomposition with respect to $\dot{\sigma}$.

Lemma 3.12. *With the above notations, let $\mathfrak{l}(\dot{\mathfrak{s}}, \mu, \varepsilon_m)$ be a realization of \mathfrak{g} adapted to $(\sigma', \sigma, \mathfrak{h})$ and $\dot{\mathfrak{t}}$ a μ -stable maximal abelian subspace of $\dot{\mathfrak{k}} = (\dot{\mathfrak{s}}_{\mathbb{R}})^{\dot{\sigma}}$. Then $Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}})$ is a $\dot{\sigma}$ stable subalgebra of $\dot{\mathfrak{s}}_{\mathbb{R}}$ of the form $Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}}) = \dot{\mathfrak{t}} \oplus \dot{\mathfrak{a}}$, where $\dot{\mathfrak{a}} \subset \dot{\mathfrak{p}}$ and $\mathfrak{h}_{\mathbb{R}}'' = (\dot{\mathfrak{t}})^{\mu} \otimes 1 \oplus (\dot{\mathfrak{a}})^{\mu} \otimes 1$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}''$.*

Proof. Given a finite-dimensional semisimple Lie algebra $\dot{\mathfrak{s}}$ over \mathbb{C} , by [7, Proposition 6.60], $Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}})$ is a $\dot{\sigma}$ -stable Cartan subalgebra of $\dot{\mathfrak{s}}_{\mathbb{R}}$ of the form $Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}}) = \dot{\mathfrak{t}} \oplus \dot{\mathfrak{a}}$, with $\dot{\mathfrak{a}} \subset \dot{\mathfrak{p}}$. The complex Lie algebra $\mathcal{Z} = (Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}}))_{\mathbb{C}}$ being a Cartan subalgebra of $\dot{\mathfrak{s}}$, is a μ -stable reductive subalgebra of $\dot{\mathfrak{s}}$. Consider the infinite abelian subalgebra $\mathfrak{z}'' = \mathfrak{l}''(\mathcal{Z}, \mu, \varepsilon_m)$ of \mathfrak{g}'' . If $x \in \mathcal{Z} \subset \dot{\mathfrak{s}}$,

$$\sigma(x \otimes t^s) = (\dot{\sigma} \otimes 1)(x \otimes t^s) = \dot{\sigma}(x) \otimes t^s.$$

As $Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}})$ is stable under the linear involution $\dot{\sigma}$, $\dot{\sigma}(x) \in \mathcal{Z}$ for $x \in \mathcal{Z}$. Hence \mathfrak{z}'' is σ -stable and consequently we have $\mathfrak{z}'' = (\mathfrak{z}'' \cap \mathfrak{l}''(\dot{\mathfrak{p}}_{\mathbb{C}}, \mu, \varepsilon_m)) \oplus (\mathfrak{z}'' \cap \mathfrak{l}''(\dot{\mathfrak{k}}_{\mathbb{C}}, \mu, \varepsilon_m))$. But by [12, Proposition 5.1], the semisimple elements of $\mathfrak{z}'' \cap \mathfrak{l}''(\dot{\mathfrak{p}}_{\mathbb{C}}, \mu, \varepsilon_m)$ are contained in $(\dot{\mathfrak{a}})^{\mu} \otimes 1$ and the semisimple elements of $\mathfrak{z}'' \cap \mathfrak{l}''(\dot{\mathfrak{k}}_{\mathbb{C}}, \mu, \varepsilon_m)$ are contained in $(\dot{\mathfrak{t}})^{\mu} \otimes 1$. Hence an element $x \in \mathfrak{z}'' \cap \mathfrak{g}_{\mathbb{R}}''$ is semisimple if and only if $x \in (\dot{\mathfrak{t}})^{\mu} \otimes 1 \oplus (\dot{\mathfrak{a}})^{\mu} \otimes 1$. Thus $(\dot{\mathfrak{t}})^{\mu} \otimes 1 \oplus (\dot{\mathfrak{a}})^{\mu} \otimes 1$ is a semi-simple abelian subalgebra of $\mathfrak{g}_{\mathbb{R}}''$. The lemma will now follow if we prove that $((\dot{\mathfrak{t}})^{\mu} \otimes 1 \oplus (\dot{\mathfrak{a}})^{\mu} \otimes 1)_{\mathbb{C}} = ((\dot{\mathfrak{t}} \oplus \dot{\mathfrak{a}})^{\mu} \otimes 1)_{\mathbb{C}}$ is a Cartan subalgebra of \mathfrak{g}'' .

From the definition of the action of μ on the Lie algebra $\mathfrak{l}(\dot{\mathfrak{s}}, Id, 1)''$ it clearly follows that,

$$((\dot{\mathfrak{t}} \oplus \dot{\mathfrak{a}})^\mu \otimes 1)_{\mathbb{C}} = (\dot{\mathfrak{t}} \oplus \dot{\mathfrak{a}})^\mu \otimes \mathbb{C} = ((\dot{\mathfrak{t}} \oplus \dot{\mathfrak{a}}) \otimes \mathbb{C})^\mu = (Z_{\dot{\mathfrak{s}}_{\mathbb{R}}}(\dot{\mathfrak{t}}) \otimes \mathbb{C})^\mu = \mathcal{Z}^\mu.$$

But \mathcal{Z} is a Cartan subalgebra of $\dot{\mathfrak{s}}$ and it is known from [6, Chapter 8], that \mathcal{Z}^μ is a Cartan subalgebra of $\mathfrak{g}'' = \mathfrak{l}(\dot{\mathfrak{s}}, \mu, \epsilon_m)''$, whenever \mathcal{Z} is a Cartan subalgebra of $\dot{\mathfrak{s}}$. Hence the claim. \blacksquare

3.13. Let $\mathfrak{g}_{\mathbb{R}}$ be an almost compact noncompact real form of \mathfrak{g} corresponding to the SI2 σ' , $\mathfrak{l}(\dot{\mathfrak{s}}, \mu, \epsilon_m)$ a realization of \mathfrak{g} adapted to σ' , σ a Cartan involution associated to σ' and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ with respect to σ . Let $\mathfrak{h}_{\mathbb{R}}'' = (\dot{\mathfrak{t}})^\mu \otimes 1 \oplus (\dot{\mathfrak{a}})^\mu \otimes 1$, with $(\dot{\mathfrak{t}})^\mu \subset \dot{\mathfrak{k}}$ and $(\dot{\mathfrak{a}})^\mu \subset \dot{\mathfrak{p}}$ be a σ -stable Cartan subalgebra of an almost compact real form $\mathfrak{g}_{\mathbb{R}}'' = \mathfrak{l}(\dot{\mathfrak{s}}_{\mathbb{R}}, \mu, \epsilon_m)''$. $\mathfrak{h}_{\mathbb{R}}''$ is said to be maximally compact if $\dim (\dot{\mathfrak{t}})^\mu$ is as large as possible. It clearly follows from paragraph 3.11 and Lemma 3.12, that all the real roots of a maximally compact Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of an almost compact real form $\mathfrak{g}_{\mathbb{R}}$ have real eigenvalues on $(\dot{\mathfrak{a}})^\mu \otimes 1$ and imaginary eigenvalues on $(\dot{\mathfrak{t}})^\mu \otimes 1$. A real root is called σ -real if it takes real values on $\mathfrak{h}_{\mathbb{R}}$ (i.e., vanishes on $(\dot{\mathfrak{t}})^\mu \otimes 1$), σ -imaginary if it takes purely imaginary values on $\mathfrak{h}_{\mathbb{R}}$ (i.e., vanishes on $(\dot{\mathfrak{a}})^\mu \otimes 1$) and σ -complex otherwise. For any $\alpha \in \Delta^{re}$, let $X_\alpha \in \mathfrak{g}_\alpha$. Then

$$[H, \sigma X_\alpha] = \sigma[\sigma^{-1}H, X_\alpha] = \alpha(\sigma^{-1}H)\sigma X_\alpha.$$

Hence $\sigma\alpha(H) = \alpha(\sigma^{-1}H)$ is a root. If α is σ -imaginary, then $\sigma\alpha = \alpha$. In this case, \mathfrak{g}_α is σ -stable, and we have $\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{k}) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{p})$. Since \mathfrak{g}_α is 1-dimensional, $\mathfrak{g}_\alpha \subset \mathfrak{k}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}$. A σ -imaginary real root α is said to be *compact* if $\mathfrak{g}_\alpha \subset \mathfrak{k}$, *noncompact* if $\mathfrak{g}_\alpha \subset \mathfrak{p}$.

3.14. Let $\alpha \in \Delta^{re}$ be σ -real. Since $\sigma\alpha = -\alpha$, $\omega'(\alpha) = -\alpha$ and $\sigma\sigma'$ is a $\text{Int}(\mathfrak{g})$ -conjugate of ω' , therefore $\sigma'H_\alpha = H_\alpha$. Consequently $H_\alpha \in \mathfrak{p} \subset \mathfrak{g}_{\mathbb{R}}$ is $\text{ad}_{\mathfrak{g}}$ -diagonalizable with real eigenvalues. For $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and $H \in \mathfrak{h}^{\sigma'}$, we have,

$$[H, \sigma'X_{\pm\alpha}] = \sigma'[\sigma'(H), X_{\pm\alpha}] = \pm\alpha(\sigma'(H))\sigma'X_{\pm\alpha} = \pm\alpha(H)\sigma'X_{\pm\alpha}.$$

Therefore $\sigma'(\mathfrak{g}_{\pm\alpha}) \subset \mathfrak{g}_{\pm\alpha}$. Fixing $E_\alpha \in \mathfrak{g}_\alpha$, $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that $B(E_\alpha, E_{-\alpha}) = 1$, we get $[E_\alpha, E_{-\alpha}] = H_\alpha$. Since for $\alpha \in \Delta^{re}$, $\dim \mathfrak{g}_{\pm\alpha} = 1$, the fact that σ' is an involution implies that $\sigma'(E_{\pm\alpha}) = E_{\pm\alpha}$ or $\sigma'(E_{\pm\alpha}) = -E_{\pm\alpha}$.

Case 1: If $\sigma'(E_{\pm\alpha}) = E_{\pm\alpha}$, then $\mathbb{R}E_\alpha \oplus \mathbb{R}H_\alpha \oplus \mathbb{R}E_{-\alpha} \subset \mathfrak{g}_{\mathbb{R}}$. By suitably scaling E_α and $E_{-\alpha}$ we can find $E_\alpha \in \mathfrak{g}_\alpha$, $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}^{\sigma'}$ such that

$$[H_\alpha, E_\alpha] = 2E_\alpha, \quad [H_\alpha, E_{-\alpha}] = -2E_{-\alpha}, \quad [E_\alpha, E_{-\alpha}] = H_\alpha.$$

From the definition of the Cartan semi-involution and the Cartan involution σ adapted to σ' it now clear that $\sigma(E_\alpha) = -E_{-\alpha}$ and $\sigma(E_{-\alpha}) = -E_\alpha$. Hence $E_\alpha - E_{-\alpha} \in \mathfrak{k}$. Corresponding to such a σ -real root α define an automorphism D_α^r as follows:

$$D_\alpha^r = \text{Ad}(\exp \frac{\pi}{4}(-E_{-\alpha} - E_\alpha)).$$

Following similar calculations as done in [7, Proposition 6.52], we get $D_\alpha^r(H_\alpha) = (E_\alpha - E_{-\alpha})$.

Case 2: If $\sigma'(E_{\pm\alpha}) = -E_{\pm\alpha}$, then $\mathbb{R}iE_\alpha \oplus \mathbb{R}H_\alpha \oplus \mathbb{R}iE_{-\alpha} \subset \mathfrak{g}_\mathbb{R}$. By suitably scaling E_α and $E_{-\alpha}$ we can find $E_\alpha \in \mathfrak{g}_\alpha$, $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}^{\sigma'}$ such that

$$[H_\alpha, iE_\alpha] = 2iE_\alpha, \quad [H_\alpha, iE_{-\alpha}] = -2iE_{-\alpha}, \quad [iE_\alpha, iE_{-\alpha}] = -H_\alpha.$$

From the definition of the Cartan semi-involution and the Cartan involution σ adapted to σ' it now clear that $\sigma(E_\alpha) = E_{-\alpha}$ and $\sigma(E_{-\alpha}) = E_\alpha$. Hence $i(E_\alpha + E_{-\alpha}) \in \mathfrak{k}$. Corresponding to such a σ -real root α define an automorphism D_α^{im} as follows:

$$D_\alpha^{im} = \text{Ad}(\exp i\frac{\pi}{4}(E_{-\alpha} - E_\alpha)).$$

Similar calculations as in [7, Proposition 6.52], show that $D_\alpha^{im}(H_\alpha) = i(E_\alpha + E_{-\alpha})$.

Given a σ -real root α , $D_\alpha^r(H_\alpha)$ and $D_\alpha^{im}(H_\alpha)$ being the image of semisimple elements are semisimple. Hence, from above discussion it follows that,

- if σ' fixes $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, then D_α^r increases the dimension of $(\mathfrak{t})^\mu \otimes 1$ by 1;
- if $-\sigma'$ fixes $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, then D_α^{im} increases the dimension of $(\mathfrak{t})^\mu \otimes 1$ by 1.

Now replacing d_α by D_α^r and D_α^{im} appropriately, the same proof as [1, Proposition 3.8] shows :

Lemma 3.15. *A σ -stable Cartan subalgebra $\mathfrak{h}_\mathbb{R}''$ of $\mathfrak{g}_\mathbb{R}''$ is maximally compact if and only if there are no σ -real real roots in $\mathfrak{h}_\mathbb{R}''$.¹*

Remark 3.16. As a consequence, we have $\mathfrak{h}_\mathbb{R} = \mathfrak{h} \cap \mathfrak{g}_\mathbb{R} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$

with
$$\mathfrak{h} \cap \mathfrak{p} = \bigoplus_{\beta \in \Delta^{\text{re}}} \mathbb{R}(p_\beta - p_{\sigma\beta}),$$

and
$$\mathfrak{h} \cap \mathfrak{k} = \left(\bigoplus_{\alpha \in \Delta^{\text{re}}} i \mathbb{R}p_\alpha \right) \oplus \left(\bigoplus_{\beta \in \Delta^{\text{re}}} \mathbb{R}(p_\beta + p_{\sigma\beta}) \right) (\text{modulo the center}),$$

where $(p_\gamma)_{\gamma \in \Delta^{\text{re}}} \subset \mathfrak{h}$ is the dual basis of the real roots. Clearly $(\mathfrak{h} \cap \mathfrak{k}) \oplus i(\mathfrak{h} \cap \mathfrak{p})$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$.

¹An independent proof of Lemma 3.15 is given in [14, 2.6 iii]

4 Vogan Diagrams

4.1. Let $\mathfrak{g}_{\mathbb{R}}$ be an almost compact real form of \mathfrak{g} and σ' be a semi-involution of second type associated to $\mathfrak{g}_{\mathbb{R}}$. Let σ be a Cartan involution adapted to σ' and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{h}_{\mathbb{R}} = (\mathfrak{i})^{\mu} \otimes 1 \oplus (\mathfrak{a})^{\mu} \otimes 1 \oplus \text{Ric} \oplus \text{Rid}$ be a maximally compact Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. By Lemma 3.15, $\mathfrak{h}_{\mathbb{R}}$ does not have any σ -real roots. Choose a positive system Δ^+ for $\Delta(\mathfrak{g}, \mathfrak{h})$, built from a basis of $i((\mathfrak{i})^{\mu} \otimes 1)$ followed by a basis of $(\mathfrak{a})^{\mu} \otimes 1$. Since $\sigma|_{i(\mathfrak{i})^{\mu} \otimes 1} = Id$, $\sigma|_{(\mathfrak{a})^{\mu} \otimes 1} = -Id$ and $\mathfrak{h}_{\mathbb{R}}$ contains no σ -real roots, $\sigma(\Delta^+) = \Delta^+$. Thus σ fixes the σ -imaginary roots and permutes in 2-cycles the σ -complex roots.

A Vogan diagram of the triple $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}, \Delta^+)$, is a Dynkin diagram of Δ^+ with the 2-element orbits of σ so labeled and with the one element orbit painted or not, accordingly as the corresponding σ -imaginary simple root is noncompact or compact. In addition to this, the underlying Dynkin has numerical labels as given in Figure 1(Section 1).

4.2. The Cartan involution σ , is an involution of first kind. Let the realization $\mathfrak{l}(\dot{\mathfrak{s}}, \zeta, \varepsilon_m)$ of \mathfrak{g} be adapted to σ . By paragraph 3.9, σ is of the form $\rho^j \exp i\pi \text{ad}(h_0)$, for $h_0 \in \mathfrak{h}_{\mathbb{Z}}^{\rho}$, $j \in \mathbb{Z}_2$, where ρ a diagram automorphism of $S(A)$. Then the base $\Pi = (\alpha_j)_{j \in [0, l]}$ can be chosen such that either $h_0 = p_j \in \mathfrak{h}$, for some $j \in [0, l]$ or $h_0 = \overset{\circ}{p}_j \in \overset{\circ}{\mathfrak{h}} \subset \mathfrak{h}$, for some $j \in [1, l]$, where the set $(p_j)_{j \in [0, l]} \subset \mathfrak{h}$ satisfies the property that $\alpha_k(p_j) = \delta_{k,j}$ for $k, j \in [0, l]$ (cf. 2.5) and $(\overset{\circ}{p}_j)_{j \in [1, l]}$ is a dual basis of the base $(\alpha_j)_{j \in [1, l]}$ of $\overset{\circ}{\mathfrak{g}} \subset \mathfrak{g}$. Since $\alpha_k(p_j) = \delta_{k,j}$ for $k, j \in \{0, 1, \dots, l\}$, it can be easily seen that,

$$\exp i\pi \text{ad}(p_k)e_j = (-1)^{\delta_{k,j}} e_k, \quad \text{for } j \in \{0, 1, \dots, l\}. \quad (4.1)$$

If the realization $\mathfrak{l}(\dot{\mathfrak{s}}, \zeta, \varepsilon_m)$ of \mathfrak{g} is adapted to $\sigma = \exp i\pi \text{ad}(p_0)$, then it follows from the definition of adapted realization and discussions following Proposition 3.10 that,

$$\begin{aligned} \text{for } m = 2, \quad \dot{\mathfrak{s}} &= \dot{\mathfrak{g}} & \zeta \otimes 1(e_0) &= -e_0, & \zeta \otimes 1(e_j) &= e_j, \text{ for } j \neq 0, \\ \text{for } m = 6, \quad \dot{\mathfrak{s}} &= \dot{\mathfrak{g}} \times \dot{\mathfrak{g}} & \zeta^3 \otimes 1(e_0) &= -e_0, & \zeta \otimes 1(e_j) &= e_j, \text{ for } j \neq 0. \end{aligned}$$

By Lemma 3.12, \mathfrak{h}^{ζ} is the Cartan subalgebra of $\mathfrak{l}(\dot{\mathfrak{s}}, \zeta, \varepsilon_m)''$ and $\exp i\pi \text{ad}(p_0)|_{\mathfrak{h}^{\zeta}} = Id = (\zeta \otimes 1)|_{\mathfrak{h}^{\zeta}}$. Since $\{e_j, f_j\}_{j \in [0, l]}$ and \mathfrak{h}^{ζ} generate \mathfrak{g} , we get:

$$\exp i\pi \text{ad}(p_0) = \zeta \otimes 1, \text{ for } m = 2, \quad \text{and} \quad \exp i\pi \text{ad}(p_0) = \zeta^3 \otimes 1, \text{ for } m = 6.$$

Also we have, $\alpha_k(\overset{\circ}{p}_j) = \delta_{k,j}$ for $k, j \in \{1, \dots, l\}$ and $\alpha_0(\overset{\circ}{p}_j) = -a_j$ for $j \in \{1, \dots, l\}$. Since $p_0 = d$ and $p_j = \overset{\circ}{p}_j + a_j d$, for $j \in [1, l]$, using Equation (4.1) we get:

$$\begin{aligned} \exp i\pi \text{ad}(\overset{\circ}{p}_k)e_j &= (-1)^{\delta_{k,j}} e_k, & \text{for } k, j \in \{1, \dots, l\}, \\ \exp i\pi \text{ad}(\overset{\circ}{p}_k)e_0 &= (-1)^{a_k} e_0, & \text{for } k \in \{1, \dots, l\}, \end{aligned} \quad (4.2)$$

Thus it follows from Eqn (4.1) and Eqn (4.2) that for $k \in \{1, 2, \dots, l\}$,

$$\begin{aligned} \exp i\pi \text{ad}(p_k) &= \exp i\pi \text{ad}(\overset{\circ}{p}_k), & \text{whenever } a_k \text{ is even for } k \in \{1, \dots, l\}, \\ \exp i\pi \text{ad}(p_k) &= (\exp i\pi \text{ad}(\overset{\circ}{p}_k)\zeta) \otimes 1, & \text{whenever } a_k \text{ is odd for } k \in \{1, \dots, l\}. \end{aligned} \quad (4.3)$$

From the symmetry of the diagram it is clear that for $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$, the involution of first type σ , is of the form $\rho \exp i\pi \text{ad}(p_j)$, $j \in [1, l]$, only when $k = 2$. In this case $\mathfrak{l}(\dot{\mathfrak{g}}, \zeta, \varepsilon_{2k}) = \mathfrak{l}(\dot{\mathfrak{g}}, \mu \exp \frac{i\pi}{2} \text{ad}(p_{\rho(0)}), \varepsilon_{2k})$ is an adapted realization of \mathfrak{g} where $p_{\rho(0)}^\circ$ is the dual of $\rho(\alpha_0)$ in $(\dot{\mathfrak{h}})^\mu$ and one gets that,

$$\begin{aligned} \exp i\pi \text{ad}(p_j) &= \exp i\pi \text{ad}(\overset{\circ}{p}_j), & \text{whenever } a_j \text{ is even for } j \in \{1, \dots, l\}, \\ \exp i\pi \text{ad}(p_j) &= (\exp i\pi \text{ad}(\overset{\circ}{p}_j)\zeta^2) \otimes 1, & \text{whenever } a_j \text{ is odd for } j \in \{1, \dots, l\}. \end{aligned} \quad (4.4)$$

4.3. It follows from Eqn(4.1) that the Vogan diagram associated to the involution $\exp i\pi \text{ad}(p_j)$, $j \in [0, l]$ has exactly one painted vertex, namely the i^{th} vertex. By Eqn(4.2), the Vogan diagrams associated to the involution $\exp i\pi \text{ad}(\overset{\circ}{p}_j)$, $j \in [1, l]$ have exactly one painted vertex, namely the j^{th} vertex, if a_j is even and exactly two painted vertices, namely the j^{th} and the 0^{th} vertices, if a_j is odd. Note that σ is of the form ρH for $\rho \neq Id$, only when \mathfrak{g} is of type $A_{2l-1}^{(2)}$ or $D_{l+1}^{(2)}$. In both the cases α_0 is a σ -complex root and hence the interior automorphism H is of the form $\exp i\pi \text{ad}(p_j)$, for some $j \in [1, l]$ such that $\rho(\alpha_j) = \alpha_j$.

4.4. Equivalence of Vogan diagrams: In [7, Chapter VI, Ex.18], an operation $R[j]$ on the Vogan diagram of a simple finite dimensional Lie algebra $\overset{\circ}{\mathfrak{g}}$ is defined as follows:

$R[j]$ acts on the base $\overset{\circ}{\pi}$ of $\overset{\circ}{\mathfrak{g}}$ by reflection corresponding to the noncompact simple root α_j . As a consequence of $R[j]$, the colour of α_j and all vertices not adjacent to α_j remain unchanged; if α_k is joined to α_j by a double edge and α_j is the smaller root, then the colour of α_k remains unchanged and if α_k is joined to α_j by single or triple lines then the colour of α_k is reversed.

Let the following operations generate equivalence relation between Vogan diagrams:

1. Application of a diagram automorphism of the Dynkin diagram $S(A)$.
2. Application of a sequence of $R[j]$ s for $j \in \{1, 2, \dots, l\}$ if the j^{th} vertex is coloured.

Given a Lie algebra $\mathfrak{g} = \mathfrak{l}(\dot{\mathfrak{g}}, \mu, \varepsilon_k)$, the set of involutions $\{\exp i\pi \text{ad}(\overset{\circ}{p}_k)\}_{k \in [1, l]}$ is a subset of $\text{Aut}((\dot{\mathfrak{g}})^\mu)$. Observe that the base of the twisted affine Kac-Moody Lie algebra \mathfrak{g} is given by $\overset{\circ}{\pi} \cup \{\alpha_0\}$, where $\alpha_0 = \delta - \theta$, for \mathfrak{g} not of type $A_{2l}^{(2)}$, and $\alpha_0 = \delta - 2\theta$, for \mathfrak{g} of type $A_{2l}^{(2)}$, $\theta \in \overset{\circ}{\Delta}_+$; $\overset{\circ}{\pi}$ is a base of $(\dot{\mathfrak{g}})^\mu$ and $\overset{\circ}{W} \ltimes T$ is the Weyl group of \mathfrak{g} (cf. Section 1) with $w(\delta) = \delta$ for $w \in \overset{\circ}{W}$. Hence, if the compact real forms of $\overset{\circ}{\mathfrak{g}}$ corresponding to the involutions $\exp i\pi \text{ad}(\overset{\circ}{p}_k)$ and $\exp i\pi \text{ad}(\overset{\circ}{p}_j)$ are isomorphic via an isomorphism induced by $\overset{\circ}{W}$, then $\overset{\circ}{W} \subset W$ induces an

isomorphism between the corresponding almost compact real forms of \mathfrak{g} . As a consequence we get the second equivalence relation as defined above.

The following are some examples of equivalent Vogan diagrams.

Example 1. $D_{l+1}^{(2)}$ ($l \geq 2$)

$$\begin{array}{ccc} \begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 1 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} & \xleftrightarrow{\text{equivalence relation 1}} & \begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 1 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 1 \\ \bullet \\ \alpha_l \end{array} \\ \sigma = \exp i\pi \text{ad}(p_0) & & \sigma = \exp i\pi \text{ad}(p_l) \end{array}$$

Example 2. $A_{2l-1}^{(2)}$

$$\begin{array}{ccc} \begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} & & \begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} \\ | & & | \\ \begin{array}{c} 1 \\ \bullet \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_i \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} & \xleftrightarrow{R[l-1] \dots R[2] R[1]} & \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_i \end{array} - \dots - \begin{array}{c} 2 \\ \bullet \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} \\ \sigma = \exp i\pi \text{ad}(p_1^\circ) & \text{equivalence relation 2} & \sigma = \exp i\pi \text{ad}(p_{l-1}^\circ) \end{array}$$

Example 3. $A_{2l-1}^{(2)}$

$$\begin{array}{ccc} \begin{array}{c} \alpha_0 \\ 1 \\ \circ \\ \alpha_1 \end{array} & & \begin{array}{c} \alpha_0 \\ 1 \\ \circ \\ \alpha_1 \end{array} \\ \updownarrow & & \updownarrow \\ \begin{array}{c} \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \bullet \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_4 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_5 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_6 \end{array} & \xleftrightarrow{R[4] R[3] R[4] R[2] R[4] R[3]} & \begin{array}{c} \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \bullet \\ \alpha_4 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_5 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_6 \end{array} \\ \text{Eq.rel 2} & & \end{array}$$

Example 4. $A_{2l-1}^{(2)}$

$$\begin{array}{ccc} \begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} & & \begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} \\ | & & | \\ \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \bullet \\ \alpha_2 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} & \xleftrightarrow{\text{Eq.rel 1}} & \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \bullet \\ \alpha_2 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} \\ & & \xleftrightarrow{R[1]} \\ & & \text{Eq.rel 2} \end{array}$$

In Example 4, “the equivalence relation 2” gives $r_1(\alpha_2) = \alpha_2 + \alpha_1$ and $r_1(\alpha_0) = \alpha_0$ and hence the final diagram is a consequence of the fact that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Remark 4.5. As a consequence of paragraph 4.2 and the equivalence relation between the Vogan diagrams, there exists a base Π' of a twisted affine Kac-Moody Lie algebra \mathfrak{g} with corresponding positive root system Δ' such that the Vogan diagram associated to $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}, \Delta')$ has at most two painted simple roots. Hence every Vogan diagram is equivalent to one with at most two painted simple roots.

Definition. An abstract Vogan diagram is an irreducible abstract Dynkin diagram of a twisted affine Kac-Moody Lie algebra \mathfrak{g} indicated with the following additional structures:

1. A diagram automorphism of order 1 or 2, indicated by labeling the 2-element orbits.
2. A subset of the 1-element orbits, indicated by painting the vertices corresponding to the members of the subset.
3. The vertices of the Dynkin diagram have enumeration as given in Figure 1.

Every Vogan diagram is of course an abstract Vogan diagram.

Theorem 4.6. *Let an abstract Vogan diagram for a twisted affine Kac-Moody Lie algebra be given. Then there exists an almost compact real form of a twisted affine Kac-Moody Lie algebra such that the given diagram is the Vogan diagram of this almost compact real form.*

Proof. It is known from [6], that a GCM A can be uniquely associated to Dynkin diagram $S(A)$ and its enumeration of vertices. By [6, Proposition 1.1], there exists a unique up-to-isomorphism realization for every GCM A . Following [6, Sect. 1.3], one can associate with A , a Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$.

By Proposition 3.7, there exists a one to one correspondence between involutions of first kind σ and the pairs (σ', \mathfrak{h}) formed of a semi-involution of second type σ' and a maximally compact Cartan subalgebra for σ' . Thus if the involution of $\mathfrak{g}(A)$ of the first kind σ , can be extracted from the additional structural information superimposed on the Vogan diagram, then one can associate to the Vogan diagram an almost compact real form of a twisted affine Kac-Moody Lie algebra. We shall now prescribe an algorithm to associate an involution of the first kind σ to a given Vogan diagram.

Let $V(A)$ denote a Vogan diagram of $\mathfrak{g}(A)$. By Remark 4.5, $V(A)$ is equivalent to a Vogan diagram of $\mathfrak{g}(A)$ with at most two painted simple roots.

1. If $V(A)$ has no painted simple roots and no 2-element orbits, then it corresponds the compact form $\mathfrak{u}(A)$ of $\mathfrak{g}(A)$ and $\sigma = Id$ in this case.
2. Suppose $V(A)$ contains no 2-element orbit. Then:
 - $\sigma = \exp i\pi \text{ad}(p_j)$, for $j \in [0, l]$, if only the j^{th} vertex is painted.
 - $\sigma = \exp i\pi \text{ad}(p_j^\circ)$, for $j \in [1, l]$, if a_j is odd and the 0^{th} and j^{th} vertices are painted.
3. Suppose $V(A)$ contains 2-element orbits. If ρ denotes the Dynkin diagram automorphism in $V(A)$ then from §4.3 it follows that:
 - $\sigma = \rho \exp i\pi \text{ad}(p_i)$, for $i \in [1, l]$, if only the i^{th} vertex is painted.
 - $\sigma = \rho$, if no vertices are painted.

The association of σ with $V(A)$ thus completes the proof of the theorem. ■

Theorem 4.7. *Two almost compact real forms of a twisted affine Kac-Moody Lie algebra \mathfrak{g} having equivalent Vogan diagram are isomorphic.*

Remark. Owing to the definition of the equivalence relation between Vogan diagrams, to prove the theorem, it suffices to show that two almost compact real forms having the same Vogan diagram are isomorphic.

Proof. Let \mathfrak{g}_1 and \mathfrak{g}_2 be two almost compact real forms of \mathfrak{g} having the same Vogan diagram. As they both have the same Dynkin diagram with the same enumeration on the vertices, the same generalized Cartan matrix A is associated with both \mathfrak{g}_1 and \mathfrak{g}_2 . Thus the unique twisted Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the complexification of \mathfrak{g}_1 and \mathfrak{g}_2 . By Proposition 3.6 there exists Cartan semi-involutions ϑ_1, ϑ_2 adapted to \mathfrak{g}_1 and \mathfrak{g}_2 respectively. Let $\mathfrak{u}_j = \mathfrak{g}^{\vartheta_j}$, for $j = 1, 2$, be the corresponding compact real forms of \mathfrak{g} . Let σ be the involution represented by the Vogan diagram. Then for $j = 1, 2$, $\sigma|_{\mathfrak{u}_j} = \varpi_j$ is the corresponding Cartan involution on \mathfrak{u}_j . Since by [11, Theorem 4.6] all Cartan semi-involutions are conjugate by $\text{Int}(\mathfrak{g})$, there exists $x \in \text{Int}(\mathfrak{g})$ such that $x\vartheta_1x^{-1} = \vartheta_2$ and consequently $x.\mathfrak{u}_1 = \mathfrak{u}_2$. As $x.\mathfrak{g}_1$ is isomorphic to \mathfrak{g}_2 , without loss of generality we may assume from the outset that $\mathfrak{u}_1 = \mathfrak{u}_2 = \mathfrak{u}$ and we have

$$\varpi_j(\mathfrak{u}) = \mathfrak{u}, \quad \text{for } j = 1, 2.$$

Let $\mathfrak{h}_1 = \mathfrak{t}_1 \oplus \mathfrak{a}_1$ and $\mathfrak{h}_2 = \mathfrak{t}_2 \oplus \mathfrak{a}_2$ be the Cartan decompositions of the Cartan subalgebras of \mathfrak{g}_1 and \mathfrak{g}_2 respectively, where \mathfrak{t}_j and \mathfrak{a}_j , for $j = 1, 2$, are respectively the $+1$ and -1 eigenspaces of σ in \mathfrak{h}_j . Consequently, for $j = 1, 2$, $\mathfrak{t}_j \oplus i\mathfrak{a}_j$ is a maximal abelian subspace of \mathfrak{u} and by Remark 3.16, $\mathfrak{t}_j \oplus i\mathfrak{a}_j$, is a maximally compact Cartan subalgebra of \mathfrak{u} . Hence by [11, Proposition 4.9c], $\mathfrak{t}_1 \oplus i\mathfrak{a}_1$ and $\mathfrak{t}_2 \oplus i\mathfrak{a}_2$ are conjugate by an element $k \in \text{Int}(\mathfrak{u})$. Replacing \mathfrak{g}_2 by $k\mathfrak{g}_2$ and arguing as above we may assume that $\mathfrak{t}_1 \oplus i\mathfrak{a}_1 = \mathfrak{t}_2 \oplus i\mathfrak{a}_2$. Thus $\mathfrak{t}_1 \oplus i\mathfrak{a}_1$ and $\mathfrak{t}_2 \oplus i\mathfrak{a}_2$ have the same complexification, which is denoted by \mathfrak{h} .

Now the complexifications \mathfrak{g} and \mathfrak{h} have been aligned and the root systems are the same. Let the respective positive root systems be given by Δ_+^1 and Δ_+^2 . By paragraph 2.6 there exists an interior automorphism $s \in \text{Int}(\mathfrak{u})$ such that $s\Delta_+^2 = \Delta_+^1$. Again replacing \mathfrak{g}_2 by $s.\mathfrak{g}_2$ and repeating above argument we assume $\Delta_+^1 = \Delta_+^2 = \Delta_+$ from the outset.

Using the conjugacy of the compact real forms of \mathfrak{g} we construct in this case,

$$\begin{aligned} \mathfrak{u} = & \sum_{\alpha \in \overset{\circ}{\Delta}} \mathbb{R}iH_\alpha \oplus \sum_{\gamma \in \Delta^{re}} \mathbb{R}i(e_\gamma + f_\gamma) \oplus \sum_{\gamma \in \Delta^{re}} \mathbb{R}(e_\gamma - f_\gamma) \oplus \mathbb{R}ic \oplus \mathbb{R}id \\ & \oplus \sum_{\alpha \in \overset{\circ}{\Delta}_l} \mathbb{R}i(H_\alpha t^{kn} + H_\alpha t^{-kn}) \oplus \sum_{\alpha \in \overset{\circ}{\Delta}_l} \mathbb{R}(H_\alpha t^{kn} - H_\alpha t^{-kn}) \oplus \\ & \left(\bigoplus_{j=0}^{k-1} \left(\sum_{\alpha \in \overset{\circ}{\Delta}_s} \mathbb{R}i(h_\alpha^j t^{kn+j} + h_\alpha^j t^{-kn-j}) \oplus \sum_{\alpha \in \overset{\circ}{\Delta}_s} \mathbb{R}(h_\alpha^j t^{kn+j} - h_\alpha^j t^{-kn-j}) \right) \right), \end{aligned}$$

where for $\gamma \in \Delta^{re}$, $e_\gamma = E_{\alpha, ks}$, $f_\gamma = E_{-\alpha, -ks}$ for $\alpha \in \overset{\circ}{\Delta}_l$, $s \in \mathbb{Z}$ and $e_\gamma = E_{\alpha, ks+j}$, $f_\gamma = E_{-\alpha, -ks-j}$ for $\alpha \in \overset{\circ}{\Delta}_s$, $s \in \mathbb{Z}$ and $j \in \mathbb{Z}_k$. Here, $\{E_{\alpha, ks}, E_{-\alpha, -ks}, H_\alpha$, for $\alpha \in \overset{\circ}{\Delta}_l\}$ and $\{E_{\alpha, ks+j}, E_{-\alpha, -ks-j}, h_\alpha^j$, for $\alpha \in \overset{\circ}{\Delta}_s, j \in \mathbb{Z}_k\}$ are defined as in paragraph 2.5.

Case 1: Suppose $\mathfrak{h}_\mathbb{R}$ has no σ -complex roots. As the Vogan diagram for \mathfrak{g}_1 and \mathfrak{g}_2 are the same, the automorphisms of Δ_+ defined by ϖ_1 and ϖ_2 have the same effect on \mathfrak{h}^* . Thus,

$$\varpi_1(H) = \varpi_2(H), \quad \text{for all } H \in \mathfrak{h}, \quad (4.5)$$

$$\varpi_1(\Delta^{im}) = \varpi_2(\Delta^{im}), \quad (4.6)$$

$$\varpi_1|_{\text{Ric} \oplus \text{Rid}} = \varpi_2|_{\text{Ric} \oplus \text{Rid}}. \quad (4.7)$$

If α is a simple σ -imaginary real root then

$$\varpi_1(e_j) = e_j = \varpi_2(e_j), \quad \text{if the } j^{\text{th}} \text{ vertex is unpainted}, \quad (4.8)$$

$$\varpi_1(e_j) = -e_j = \varpi_2(e_j), \quad \text{if the } j^{\text{th}} \text{ vertex is painted}. \quad (4.9)$$

Since \mathfrak{h} and $\{e_j, f_j, j \in [0, l]\}$ generate \mathfrak{g} , it follows that $\varpi_1 = \varpi_2$ on \mathfrak{u} , hence $\mathfrak{k}_1 = \mathfrak{u}^{\varpi_1} = \mathfrak{k}_2$ and if for $j = 1, 2$, \mathfrak{p}_j denotes the -1 eigenspace of ϖ_j on \mathfrak{u} , then $\mathfrak{p}_1 = \mathfrak{p}_2$. Hence

$$\mathfrak{g}_1 = \mathfrak{k}_1 \oplus i\mathfrak{p}_1 = \mathfrak{k}_2 \oplus i\mathfrak{p}_2 = \mathfrak{g}_2.$$

Case 2: Suppose there exists σ -complex simple real roots in \mathfrak{g}_1 and \mathfrak{g}_2 . Let ρ denote the diagram automorphism of $S(A)$. In this case the σ -imaginary roots are treated as in Case 1. If for all σ -complex roots $\gamma \in \Delta^{re}$ there exists $X_\gamma \in \mathfrak{g}_\gamma, X_{\rho\gamma} \in \mathfrak{g}_{\rho\gamma}$ and constants a_γ, b_γ such that

$$\varpi_1(X_\gamma) = a_\gamma X_{\rho\gamma} \quad \text{and} \quad \varpi_2(X_\gamma) = b_\gamma X_{\rho\gamma}, \quad (4.10)$$

then the same calculations as done in [1, Theorem 5.2(Case 2)] show that for each pair of σ -complex simple roots $\gamma, \varpi\gamma$, square roots $a_\gamma^{1/2}, a_{\rho\gamma}^{1/2}, b_\gamma^{1/2}, b_{\rho\gamma}^{1/2}$ can be chosen such that $a_\gamma^{1/2}, a_{\rho\gamma}^{1/2} = 1$, and $b_\gamma^{1/2}, b_{\rho\gamma}^{1/2} = 1$. Then defining $H, H' \in \mathfrak{u} \cap \mathfrak{h}$ such that $\alpha(H) = \alpha(H') = 0$ for α σ -imaginary simple root and

$$\begin{aligned} \exp(\tfrac{1}{2}\gamma(H)) &= a_\gamma^{1/2}, & \exp(\tfrac{1}{2}\rho\gamma(H)) &= a_{\rho\gamma}^{1/2}, \\ \exp(\tfrac{1}{2}\gamma(H')) &= b_\gamma^{1/2}, & \exp(\tfrac{1}{2}\rho\gamma(H')) &= b_{\rho\gamma}^{1/2}, \end{aligned}$$

for $\gamma, \rho\gamma$ σ -complex simple roots, similar calculations as in [1, Theorem 5.2(Case 2)] show that application of the identity

$$\varpi_2 \circ \text{Ad}(\exp \tfrac{1}{2}(H - H')) = \text{Ad}(\exp \tfrac{1}{2}(H - H')) \circ \varpi_1,$$

to the σ -eigenspaces of \mathfrak{g}_1 gives an isomorphism between \mathfrak{g}_1 and \mathfrak{g}_2 .

Thus to complete the proof of the theorem we need to show the existence of $X_\gamma \in \mathfrak{g}_\gamma$, $X_{\rho_\gamma} \in \mathfrak{g}_{\rho_\gamma}$ and constants a_γ, b_γ satisfying Eqn(4.10). Observe from Figure 1, that for \mathfrak{g} of type Aff $k, k \neq 1$, σ -complex roots exist only when \mathfrak{g} of type $A_{2\ell-1}^{(2)}$ or $D_{\ell+1}^{(2)}$. If $\gamma = \alpha + ks\delta \in \Delta^{re}$ for $\alpha \in \overset{\circ}{\Delta}_l$, then $\mathfrak{g}_\gamma = \mathbb{C}e_\alpha \otimes t^{ks}$, where $e_\alpha \in (\dot{\mathfrak{g}})_\alpha$. As $e_\alpha \otimes t^{ks} \in \mathfrak{l}(\dot{\mathfrak{g}}, Id, 1)$, by [1, Theorem 5.2(Case 2)], Eqn (4.10) is satisfied in this case. However problems can arise if a σ -complex root $\gamma \in \Delta^{re}$ is a short root. Note that the following are the only short σ -complex simple roots:

$$\begin{aligned} \alpha_0, \alpha_1 = \alpha_{\rho_0}, & \quad \text{when } \mathfrak{g} \text{ of type } A_{2\ell-1}^{(2)}, \\ \alpha_0, \alpha_\ell = \alpha_{\rho_0}, & \quad \text{when } \mathfrak{g} \text{ of type } D_{\ell+1}^{(2)}. \end{aligned} \quad (4.11)$$

Let $H_{\rho_{(0)}} := \exp \frac{i\pi}{2} \text{ad}(\overset{\circ}{(p_{\rho_{(0)}})}))$. Then $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_{\rho_{(0)}}, \varepsilon_4)$ is a realization of \mathfrak{g} adapted to the involution σ , having a 2-element orbit. If $\mathfrak{g}_{\alpha_0} = \mathbb{C}X_{\alpha_0}$ and $\mathfrak{g}_{\rho(\alpha_0)} = \mathbb{C}X_{\alpha_j}$, for α_j a simple short root of \mathfrak{g} , then in the realization, $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_{\rho_{(0)}}, \varepsilon_4)$ we have,

$$\begin{aligned} X_{\alpha_0} &= (e_{-\theta^0} - e_{-\mu\theta^0}) \otimes t & \text{for } \mathfrak{g} \text{ of type } D_{\ell+1}^{(2)} \text{ and } A_{2\ell-1}^{(2)} \\ X_{\alpha_j} &= (e_{\dot{\alpha}_j} + e_{-\mu\dot{\alpha}_j}) \otimes t & \text{where } \begin{cases} j = \ell \text{ for } \mathfrak{g} \text{ of type } D_{\ell+1}^{(2)}, \dot{\alpha}_j \in \Delta(D_{\ell+1}) \\ j = 1 \text{ for } \mathfrak{g} \text{ of type } A_{2\ell-1}^{(2)}, \dot{\alpha}_j \in \Delta(A_{2\ell-1}) \end{cases} \end{aligned} \quad (4.12)$$

Since for appropriate j (as explained in Eqn(4.12)), $\varpi_1(\mathfrak{g}_{\alpha_0}) \subset \mathfrak{g}_{\alpha_j}$ and $\varpi_2(\mathfrak{g}_{\alpha_0}) \subset \mathfrak{g}_{\alpha_j}$, by Eqn(4.12) there exists constants $a_{\theta^0}^1, a_{\theta^0}^2, a_{\mu\theta^0}^1, a_{\mu\theta^0}^2$ and $b_{\theta^0}^1, b_{\theta^0}^2, b_{\mu\theta^0}^1, b_{\mu\theta^0}^2$ such that

$$\begin{aligned} \varpi_1(e_{-\theta^0} \otimes t) &= a_{\theta^0}^1 e_{\alpha_j} \otimes t + a_{\mu\theta^0}^1 e_{\mu\alpha_j} \otimes t, & \varpi_1(e_{-\mu\theta^0} \otimes t) &= a_{\theta^0}^2 e_{\alpha_j} \otimes t + a_{\mu\theta^0}^2 e_{\mu\alpha_j} \otimes t, \\ \varpi_2(e_{-\theta^0} \otimes t) &= b_{\theta^0}^1 e_{\alpha_j} \otimes t + b_{\mu\theta^0}^1 e_{\mu\alpha_j} \otimes t, & \varpi_2(e_{-\mu\theta^0} \otimes t) &= b_{\theta^0}^2 e_{\alpha_j} \otimes t + b_{\mu\theta^0}^2 e_{\mu\alpha_j} \otimes t. \end{aligned}$$

Claim: $a_{\theta^0}^1 - a_{\theta^0}^2 = a_{\mu\theta^0}^1 - a_{\mu\theta^0}^2$; $b_{\theta^0}^1 - b_{\theta^0}^2 = b_{\mu\theta^0}^1 - b_{\mu\theta^0}^2$.

Proof the claim: Recall that ϖ_1 and ϖ_2 are restrictions of the involution σ to the compact forms \mathfrak{u}_1 and \mathfrak{u}_2 adapted to \mathfrak{g}_1 and \mathfrak{g}_2 respectively. Since for $j = 1, 2$, the Cartan involution σ is adapted to both \mathfrak{g}_j and the realization $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_{\rho_{(0)}}, \varepsilon_4)$ of \mathfrak{g} is adapted to $(\mathfrak{g}_j, \sigma, \mathfrak{h})$, by [12, Proposition 3.5] there exists $\dot{\sigma}_j \in \text{Aut}(\dot{\mathfrak{g}})$ commuting with $\mu H_{\rho_{(0)}}$ such that $\varpi_j = \dot{\sigma}_j \otimes 1$ on the compact form \mathfrak{u} of $\mathfrak{l}(\dot{\mathfrak{g}}, \mu H_{\rho_{(0)}}, \varepsilon_4)$.

As $\mu H_{\rho_{(0)}}(e_{\mu\theta^0} \otimes t) = -i e_{\theta^0} \otimes t$, we have $e_{\theta^0} \otimes t = i \mu H_{\rho_{(0)}}(e_{\mu\theta^0} \otimes t)$. Hence,

$$\begin{aligned} a_{\theta^0}^2 e_{\alpha_j} \otimes t + a_{\mu\theta^0}^2 e_{\mu\alpha_j} \otimes t &= \varpi_1(e_{-\mu\theta^0} \otimes t) = \varpi_1(i \mu H_{\rho_{(0)}}(e_{-\theta^0}) \otimes t) \\ &= \dot{\sigma}_1(i(\mu H_{\rho_{(0)}}(e_{-\theta^0})) \otimes t) = i \mu H_{\rho_{(0)}}(\dot{\sigma}_1(e_{-\theta^0})) \otimes t \\ &= i \mu H_{\rho_{(0)}}((\dot{\sigma}_1(e_{-\theta^0})) \otimes t) = i \mu H_{\rho_{(0)}}((\varpi_1(e_{-\theta^0} \otimes t))) \\ &= i \mu H_{\rho_{(0)}}(a_{\theta^0}^1 e_{\alpha_j} \otimes t + a_{\mu\theta^0}^1 e_{\mu\alpha_j} \otimes t) \\ &= i^2 (a_{\theta^0}^1 e_{\mu\alpha_j} \otimes t + a_{\mu\theta^0}^1 e_{\alpha_j} \otimes t) = -a_{\theta^0}^1 e_{\mu\alpha_j} \otimes t - a_{\mu\theta^0}^1 e_{\alpha_j} \otimes t. \end{aligned}$$

Comparing coefficients we get, $a_{\theta^0}^2 = -a_{\mu\theta^0}^1$ and $a_{\mu\theta^0}^2 = -a_{\theta^0}^1$. Hence $a_{\theta^0}^1 - a_{\theta^0}^2 = -a_{\mu\theta^0}^2 + a_{\mu\theta^0}^1$ as desired. It can be similarly shown that $b_{\theta^0}^1 - b_{\theta^0}^2 = -b_{\mu\theta^0}^2 + b_{\mu\theta^0}^1$. Thus for short σ -complex simple roots $X_{\alpha_0} \in \mathfrak{g}_{\alpha_0}$ and $X_{\alpha_j} \in \mathfrak{g}_{\rho\alpha_0}$ exists such that Eqn(4.10) is satisfied. Hence the theorem. \blacksquare

Using the definition of equivalence relations between the Vogan diagrams (cf. 4.4) and [7, Figure 6.1, Figure 6.2] we give in the following table the non-equivalent Vogan diagrams of the twisted affine Kac-Moody Lie algebras corresponding to non-trivial involutions of first type. Note that owing to the equivalence relation as described in Example 1, the non-equivalent Vogan diagrams for \mathfrak{g} correspond to the following involutions of first type : $\mu \otimes 1, \exp i\pi \text{ad}(p_i)$ for $1 \leq i \leq \frac{l}{2}$, $\exp i\pi \text{ad}(p_i^\circ)$ for $1 \leq i \leq l$. Likewise, owing to the equivalence relations described in Examples 2 and 4, the non-equivalent Vogan diagrams for \mathfrak{g} correspond to the following involutions of first type : $\mu \otimes 1, \exp i\pi \text{ad}(p_l), \exp i\pi \text{ad}(p_l^\circ), \exp i\pi \text{ad}(p_j^\circ)$ for $1 \leq j \leq \frac{l}{2}$ and owing to the equivalence relations of the kind described in Example 3, the non-equivalent Vogan diagrams for \mathfrak{g} correspond to the following involutions of first type : $\rho, \rho(\mu^2 \exp i\pi \text{ad}(p_1^\circ) \otimes 1) \exp i\pi \text{ad}(p_l^\circ), \rho \exp i\pi \text{ad}(p_j^\circ)$ for $1 \leq j \leq \frac{l+1}{2}$. The non-equivalent Vogan diagrams for the twisted affine Kac-Moody Lie algebras of type $A_{2l}^{(2)}, D_{2r+1}^{(2)}, r \geq 1, D_{2r}^{(2)}, r \geq 2, E_6^{(2)}$ and $D_4^{(2)}$ are similarly studied.

It can be easily observed from the Figures 2 and 3 that the count of the number of the Vogan diagrams corresponding to non-trivial involutions of first type matches with the number of almost compact non-compact real forms of twisted affine Kac-Moody Lie algebras as given in [12], thereby suggesting the existence of a bijective correspondence between the equivalence classes of the Vogan diagrams and the isomorphism classes of the almost compact real forms of twisted affine Kac-Moody Lie algebras.

\mathfrak{g}	<i>e-Vogan diagram</i>	<i>involution of first type</i>
$A_2^{(2)}$	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} = \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array}$	$\mu \otimes 1$
	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} = \begin{array}{c} 2 \\ \bullet \\ \alpha_1 \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_1) = \exp i\pi \text{ad}(p_1)$
$A_{2l}^{(2)}$	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} \Rightarrow \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 2 \\ \circ \\ \alpha_l \end{array}$	$\mu \otimes 1$
	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} \Rightarrow \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 2 \\ \bullet \\ \alpha_i \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 2 \\ \circ \\ \alpha_l \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_i) = \exp i\pi \text{ad}(p_i),$ $1 \leq i \leq l$
$A_{2l-1}^{(2)} \ (l \geq 3)$	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \\ \\ \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} \end{array}$	$\mu \otimes 1$
	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \\ \\ \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \bullet \\ \alpha_l \end{array} \end{array}$	$(\mu \otimes 1) \exp i\pi \text{ad}(\overset{\circ}{p}_l) = \exp i\pi \text{ad}(p_l)$
	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \\ \\ \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \bullet \\ \alpha_l \end{array} \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_l)$
	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \\ \\ \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \dots - \begin{array}{c} 2 \\ \bullet \\ \alpha_i \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_i), 1 \leq i \leq \frac{l}{2}$
$A_{2l-1}^{(2)} \ (l > 2)$	$\begin{array}{c} \alpha_0 \\ 1 \\ \circ \\ \updownarrow \\ \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} \end{array}$	ρ
	$\begin{array}{c} \alpha_0 \\ 1 \\ \circ \\ \alpha_1 \\ \updownarrow \\ \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \bullet \\ \alpha_l \end{array} \end{array}$	$\rho(\mu^2 \exp i\pi \text{ad}(\overset{\circ}{p}_1) \otimes 1) \exp i\pi \text{ad}(\overset{\circ}{p}_l)$
	$\begin{array}{c} \alpha_0 \\ 1 \\ \circ \\ \alpha_1 \\ \updownarrow \\ \begin{array}{c} 2 \\ \circ \\ \alpha_2 \end{array} - \dots - \begin{array}{c} 2 \\ \bullet \\ \alpha_i \end{array} - \dots - \begin{array}{c} 2 \\ \circ \\ \alpha_{l-1} \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array} \end{array}$	$\rho \exp i\pi \text{ad}(\overset{\circ}{p}_i), 2 \leq i \leq \frac{l+1}{2}$

Figure 2: Vogan diagrams for affine Kac-Moody Lie algebras of type Aff 2

\mathfrak{g}	<i>e-Vogan diagram</i>	<i>involution of first type</i>
$D_{l+1}^{(2)} (l \geq 2)$	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 1 \\ \circ \\ \alpha_i \end{array} - \dots - \begin{array}{c} 1 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array}$	$\mu \otimes 1$
	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 1 \\ \bullet \\ \alpha_i \end{array} - \dots - \begin{array}{c} 1 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_i)\mu \otimes 1 =$ $\exp i\pi \text{ad}(p_i), 1 \leq i \leq \frac{l}{2}$
	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_1 \end{array} - \dots - \begin{array}{c} 1 \\ \bullet \\ \alpha_i \end{array} - \dots - \begin{array}{c} 1 \\ \circ \\ \alpha_{l-1} \end{array} \Rightarrow \begin{array}{c} 1 \\ \circ \\ \alpha_l \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_i), 1 \leq i \leq l$
$D_{2r+1}^{(2)} (r \geq 1)$	$\begin{array}{c} \alpha_0 \\ \circ_1 \end{array} \Leftarrow \begin{array}{c} \alpha_1 \\ \circ_1 \end{array} - \dots - \begin{array}{c} \alpha_{r-1} \\ \circ_1 \end{array} \searrow \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \alpha_r \overset{1}{\circ} \\ \begin{array}{c} \circ^1 \\ \alpha_{2r} \end{array} \Leftarrow \begin{array}{c} \circ^1 \\ \alpha_{2r-1} \end{array} - \dots - \begin{array}{c} \circ^1 \\ \alpha_{r+1} \end{array} \nearrow$	ρ
	$\begin{array}{c} \alpha_0 \\ \circ_1 \end{array} \Leftarrow \begin{array}{c} \alpha_1 \\ \circ_1 \end{array} - \dots - \begin{array}{c} \alpha_{r-1} \\ \circ_1 \end{array} \searrow \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \alpha_r \overset{1}{\bullet} \\ \begin{array}{c} \circ^1 \\ \alpha_{2r} \end{array} \Leftarrow \begin{array}{c} \circ^1 \\ \alpha_{2r-1} \end{array} - \dots - \begin{array}{c} \circ^1 \\ \alpha_{r+1} \end{array} \nearrow$	$\rho \exp i\pi \text{ad}(p_r) =$ $\rho \exp i\pi \text{ad}(\overset{\circ}{p}_r)(\mu^2 \exp i\pi \text{ad}(\overset{\circ}{p}_{2r}) \otimes 1)$
$D_{2r}^{(2)} (r \geq 2)$	$\begin{array}{c} \alpha_0 \\ \circ_1 \end{array} \Leftarrow \begin{array}{c} \alpha_1 \\ \circ_1 \end{array} - \dots - \begin{array}{c} \alpha_{r-1} \\ \circ_1 \end{array} \searrow \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \left. \right) \\ \begin{array}{c} \circ^1 \\ \alpha_{2r-1} \end{array} \Leftarrow \begin{array}{c} \circ^1 \\ \alpha_{2r-2} \end{array} - \dots - \begin{array}{c} \circ^1 \\ \alpha_{r+1} \end{array}$	ρ
$E_6^{(2)}$	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} - \begin{array}{c} 2 \\ \bullet \\ \alpha_1 \end{array} - \begin{array}{c} 3 \\ \circ \\ \alpha_2 \end{array} \Leftarrow \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \begin{array}{c} 1 \\ \circ \\ \alpha_4 \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_1)$ $= \exp i\pi \text{ad}(p_1)$
	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 3 \\ \circ \\ \alpha_2 \end{array} \Leftarrow \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \begin{array}{c} 1 \\ \bullet \\ \alpha_4 \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_4)$
	$\begin{array}{c} 1 \\ \circ \\ \alpha_0 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 3 \\ \circ \\ \alpha_2 \end{array} \Leftarrow \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \begin{array}{c} 1 \\ \bullet \\ \alpha_4 \end{array}$	$(\mu \otimes 1) \exp i\pi \text{ad}(\overset{\circ}{p}_4)$ $= \exp i\pi \text{ad}(p_4)$
	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} - \begin{array}{c} 3 \\ \circ \\ \alpha_2 \end{array} \Leftarrow \begin{array}{c} 2 \\ \circ \\ \alpha_3 \end{array} - \begin{array}{c} 1 \\ \circ \\ \alpha_4 \end{array}$	$\mu \otimes 1$
$D_4^{(3)}$	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} \Leftarrow \begin{array}{c} 1 \\ \bullet \\ \alpha_2 \end{array}$	$\exp i\pi \text{ad}(\overset{\circ}{p}_2)$
	$\begin{array}{c} 1 \\ \bullet \\ \alpha_0 \end{array} - \begin{array}{c} 2 \\ \circ \\ \alpha_1 \end{array} \Leftarrow \begin{array}{c} 1 \\ \circ \\ \alpha_2 \end{array}$	$\zeta^3 \otimes 1$

Figure 3: Vogan diagrams for affine Kac-Moody Lie algebras of type Aff 2 and 3 (contd.)

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